United Kingdom
Mathematics Trust

Senior Mathematical Challenge

Tuesday 6th November 2018

For reasons of space, these solutions are necessarily brief.

There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation:

www.ukmt.org.uk

1. C The values of the expressions are 1, 8, 81, 1024 and 15 625 respectively, so there are three odd numbers. Alternatively, note also that positive integer powers of odd and even numbers are themselves odd and even respectively.

2. C As 2018 is the product of the primes 2 and 1009, the sum of these primes is 1011.

3. D Option D shows the image after the rotation of 135° clockwise.

4. E Option A can be expressed using the difference of two squares as $2019^2 - 2014^2 = (2019 + 2014)(2019 - 2014) = 4033 \times 5$, so this answer is a multiple of 5.

Option B is $2019^2 \times 10^2$ which is $2019^2 \times 10 \times 2 \times 5$, which again is a multiple of 5.

Option C is $\frac{2020^2}{101^2}$ which equals $\left(\frac{2020}{101}\right)^2$ and this simplifies to 20^2 which is 80×5, so the answer is a multiple of 5.

Option D is $2010^2 - 2005^2$ which equals $(2010 + 2005)(2010 - 2005)$, so 4015×5 which again is a multiple of 5.

However, option E is $\frac{2015^2}{5^2}$ which equals $\left(\frac{2015}{5}\right)^2$ and this is 403^2. As the final digit of 403^2 is 9, this is not a multiple of 5.

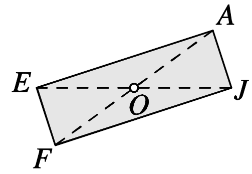
5. B Only option B is greater than four, so this is the largest.

6. B The expression $25 \times 15 \times 9 \times 5.4 \times 3.24 = 5^2 \times 3 \times 5 \times 3^2 \times \frac{54}{10} \times \frac{324}{100}$ which factorises further to $5^2 \times 3 \times 5 \times 3^2 \times \frac{2\times3^3}{2\times5} \times \frac{2^2\times3^4}{2^2\times5^2}$. All the factors of 2 and 5 in the numerator and denominator cancel to leave only the product of powers of 3. This is $3^1 \times 3^2 \times 3^3 \times 3^4$, so the expression is equal to 3^{10}.

7. A Let the radius of the circle Q be q and the radius of the circle R be r. The radius of the largest circle, P, can therefore be written as $(q + r)$. Using 'circumference = $2\pi \times$ radius', the required expression becomes $\dfrac{2\pi q + 2\pi r}{2\pi(q + r)}$ which equals $\dfrac{2\pi(q + r)}{2\pi(q + r)}$ and so has value 1.

8. B The first four terms of the geometric sequence 7^n are 7, 49, 343 and 2401. Evaluating 7^5 by considering 2401×7 shows that the tens and units digits are '07' as in the first term. Hence a cyclical pattern for the tens and units digits is formed using '07', '49', '43' and '01'. As $2018 = 504 \times 4 + 2$, then 7^{2018} must end in the second value of the cycle, so '49'.

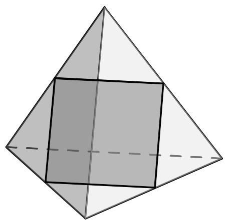
9. A Let O be the centre of the regular decagon. The decagon can be split into ten congruent isosceles triangles each with a vertex at O. Triangles AOJ and EOF are two of these ten isosceles triangles. As O is also the centre of the rectangle and the diagonals of a rectangle split its interior into four equal areas, the triangles EOA and FOJ each have the same area as triangle AOJ. The required ratio is then 4 : 10 which simplifies to 2 : 5.



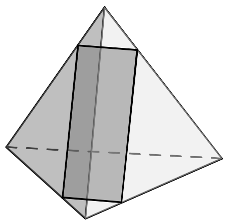
10. D In order to calculate Laura's average speed over the whole journey we can first calculate the total distance travelled and the total time taken. Using "distance in km = speed in km/h × time in h', the total distance travelled, in km, is $12 \times \frac{5}{60} + 15 \times \frac{10}{60} + 18 \times \frac{15}{60}$ which is $1 + 2.5 + 4.5$, so 8. The total time taken is 5 + 10 + 15 minutes, so $\frac{1}{2}$ hour. Laura's average speed is then $\frac{8}{1/2} = 16\,\text{km/h}$.

11. B For each equation, consider $x = 0$. Then, $y = x^4 + 1$ becomes $y = 0^4 + 1 = 1$ and so the graph with this equation passes through $(0, 1)$ rather than $(0, 0)$. For the other three equations, using $x = 0$ shows that $y = 0$ too and so their graphs all pass through the origin. Hence only one graph does not pass through the origin.

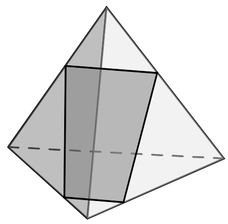
12. A When the regular tetrahedron is cut by a single plane cut, each of its four faces is cut at most once. The line along which each face is cut becomes an edge of the newly formed section. Therefore the pentagon, with five edges, cannot be formed. Each of the other four options is possible as the following examples show.



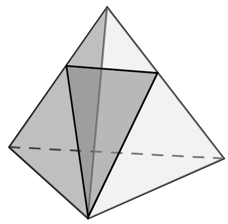
A square



A rectangle that is not a square



A trapezium

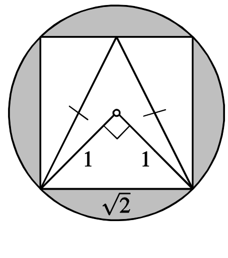


A triangle that is not equilateral

13. E Since $y = x$ and $y = mx - 4$, it follows that $x = mx - 4$ and so $4 = mx - x = (m - 1)x$; therefore $x = \frac{4}{m-1}$. In order for x to be a positive integer, $m - 1$ must be a positive factor of 4 therefore $m - 1$ can equal 1, 2 or 4 and so m can equal 2, 3 or 5. The sum of these values of m is then 10.

14. D The mean of the twelve integers is given to be 7. Therefore $1 + 3x + 5y + 8 + 9 + 11 = 12 \times 7$ so $3x + 5y = 55$. As the twelve integers are written in ascending order, the median value is half way between the 6th and 7th values, each of which is y, so the median is y. Rearranging our equation gives $y = \dfrac{55 - 3x}{5}$ and hence $3x$ must be a multiple of 5 so that the numerator can be divided exactly by 5. To ensure that the twelve integers are in ascending order the minimum value of x is 1 and the maximum value of x is 8. The only value in this interval which makes $3x$ a multiple of 5 is $x = 5$, in which case $y = \dfrac{55 - 15}{5} = 8$.

15. D The circle has radius 1, so, using Pythagoras' Theorem, the length of the side of the square is $\sqrt{2}$. The area of the shaded region is then $\pi \times 1^2 - \sqrt{2}^2$ which equals $\pi - 2$. The base of the original isosceles triangle is also one of the sides of the square and so has length $\sqrt{2}$. Its perpendicular height is also $\sqrt{2}$ as it is parallel to two of the other sides of the square. The area of the triangle is then $\frac{1}{2} \times \sqrt{2} \times \sqrt{2}$ which is 1. The required ratio is then $1 : \pi - 2$.



16. D Considering the first set of equations $4p = 3q = 2r = s$, we can see that $p < q < r < s$. Also, s must be a multiple of 2, 3 and 4. The smallest such multiple is 12 which gives $p = 3$, $q = 4$, $r = 6$ and $s = 12$. In the equation $p + 2q + 3r + 4s = k$ we then have $3 + 2 \times 4 + 3 \times 6 + 4 \times 12 = k$, so $k = 77$.

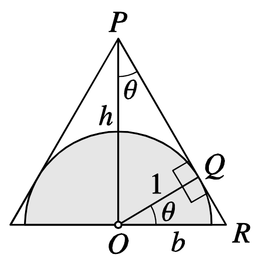
17. B Let the number of 20p coins and 50p coins Bethany has be x and y respectively. Her total number of coins is then $11 + x + y$. As the mean value of her coins is 52p, the total value of her coins is both $52 \times (11 + x + y)$ and $100 \times 11 + 20x + 50y$. The equation formed is $52(11 + x + y) = 1100 + 20x + 50y$ which becomes $572 + 52x + 52y = 1100 + 20x + 50y$ and this simplifies to $32x + 2y = 528$ or $16x + y = 264$. By rearranging we can see that $11 + x + y = 275 - 15x$. So Bethany's total number of coins is of the form $275 - 15x$, or $5 + 15(18 - x)$ which gives values that are 5 more than multiples of 15. Of the given options the only one which is not 5 more than a multiple of 15 is 40. We can also see that Bethany must have a large purse!

18. A Let the three angles P, Q and R be written in the form '$p.a$', '$q.b$' and '$r.c$' respectively, where p, q, and r are positive integers and a, b and c each represent the full decimal part of the angle. The sum of the decimal parts of the angles falls into one of three cases: (i) $a + b + c = 0$, so $p + q + r = 180$; (ii) $a + b + c = 1$, so $p + q + r = 179$; (iii) $a + b + c = 2$, so $p + q + r = 178$.

In case (i) no rounding is necessary as all three angles are integers so $P + Q + R = 180$. In case (ii), none (e.g. 'p.4', 'q.4', 'r.2'), one (e.g. 'p.5', 'q.4', 'r.1') or two (e.g. 'p.5', 'q.5', 'r.0') could be large enough for the respective integer parts to 'round up'. So $P + Q + R$ could be $179 + 0$, $179 + 1$ or $179 + 2$. In case (iii) either two (e.g. 'p.9', 'q.9', 'r.2') or three (e.g. 'p.7', 'q.7', 'r.6') of a, b and c could be large enough for the respective integer parts to 'round up'. So the total of $P + Q + R$ could be $178 + 2$ or $178 + 3$. The complete list of possible values of $P + Q + R$ is then 179, 180, 181.

19. C If the exterior angle of an m-sided polygon is $n°$ then its number of sides, m, is $\frac{360}{n}$. Similarly if the exterior angle of an n-sided polygon is $m°$, then its number of sides, n, is $\frac{360}{m}$. Both n and m must be positive integers greater than two (or no polygon can be formed) and $n \times m$ must be 360. The value of m can therefore be any of the twenty numbers 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90 or 120 and the corresponding value of n will be $\frac{360}{m}$.

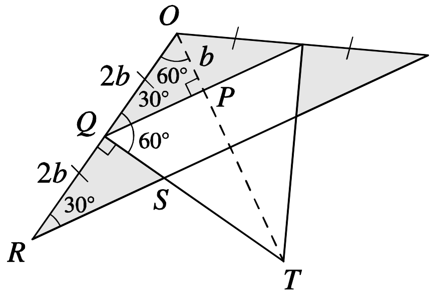
20. E Let O be the centre of the circle, P be the vertex at the top of the triangle, R the vertex at one end of the 'base' and Q be the point where PR is tangent to the semicircle, as shown. Let OP have length h and OR have length b. The line OQ is a radius of the semicircle so has length 1 and is perpendicular to PR.



As OP and OR are perpendicular, triangles ROQ and OPQ are similar and so angle ROQ and angle OPQ are equal. Triangles ROQ and OPQ can be used to find expressions for the lengths of the base and height of the original isosceles triangle. Considering triangle ROQ gives $\cos\theta = \frac{1}{b}$ so $b = \frac{1}{\cos\theta}$. Considering the triangle OPQ gives $\sin\theta = \frac{1}{h}$ so $h = \frac{1}{\sin\theta}$. Then the area of the isosceles triangle is $\frac{1}{2} \times 2b \times h = \frac{1}{\cos\theta} \times \frac{1}{\sin\theta} = \frac{1}{\sin\theta\cos\theta}$.

21. A After reflecting the graph of $y = \frac{1}{x}$ in the line $y = 1$ we have a graph whose equation is $y = 1 + (1 - \frac{1}{x})$, so $y = -\frac{1}{x} + 2$. The second reflection is in the line $y = -x$. Here, y is replaced by $-x$ and $-x$ is replaced by y. Hence the equation $y = -\frac{1}{x} + 2$ becomes $-x = -\frac{1}{-y} + 2$ which rearranges to $-(x + 2) = \frac{1}{y}$ and then to $y = \dfrac{-1}{(x + 2)}$.

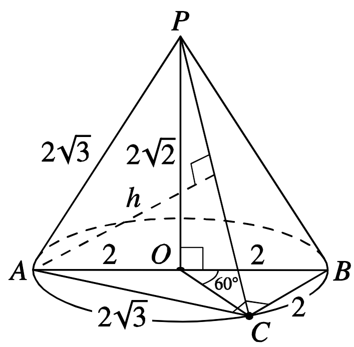
22. C As two vertices of the equilateral triangle are given to be midpoints of the equal sides of the isosceles triangle, OT is a line of symmetry. Let P be the point where OT intersects an edge of the equilateral triangle, as shown. By using the line of symmetry through O, P and T, we can see the shaded region as four grey triangles. As we are given an angle of 120° at O, and angles of an equilateral triangle are each 60°, the four grey triangles each have angles 30°, 60°, 90° and sides whose lengths are in the ratios $1 : \sqrt{3} : 2$.



Let $OP = b$. Then $OQ = QR = 2b$. Also $QP = \sqrt{3}b$ and $QS = \frac{2b}{\sqrt{3}}$ using similar triangles. The total shaded area is then $2(\frac{1}{2} \times b \times \sqrt{3}b + \frac{1}{2} \times \frac{2b}{\sqrt{3}} \times 2b) = \sqrt{3}b^2 + \frac{4b^2}{\sqrt{3}} = \frac{7b^2}{\sqrt{3}}$. Triangle QPT also has sides in the ratios $1 : \sqrt{3} : 2$ and as $QP = \sqrt{3}b$, $PT = 3b$. The area of the equilateral triangle is then $\frac{1}{2} \times 2\sqrt{3}b \times 3b = 3\sqrt{3}b^2$ which we are told is 36. So $3\sqrt{3}b^2 = 36$ and therefore $\frac{b^2}{\sqrt{3}} = 4$. Hence the area of the shaded region is $\frac{7b^2}{\sqrt{3}} = 7 \times 4 = 28$.

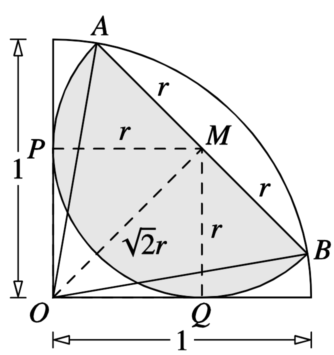
23. B As $f(x) = ax+b$, $f(f(f(x))) = a(a(ax+b)+b)+b$ which expands to give $a^3x+a^2b+ab+b$ and this can be written as $a^3x+b(a^2+a+1)$. We therefore have $a^3x+b(a^2+a+1) = 27x-52$. Equating coefficients of x gives $a^3 = 27$, so $a = 3$. Equating the constants gives $b(3^2+3+1) = -52$, so $b = -4$. So the function f is given by $f(x) = 3x - 4$. As we are given that $g(f(x)) = x$, the function g uses inverse operations to undo the two steps in the function f, in order to return to x. As the operations in f are 'multiply by 3' then 'subtract 4', the operations in g are 'add 4' then 'divide by 3'. Therefore the function g is given by $g(x) = \frac{x+4}{3} = \frac{1}{3}x + \frac{4}{3}$.

24. E As points A and C are both on the circular base with centre O, and P is directly above O, we have $PA = PC$. Applying Pythagoras' Theorem to triangle OPA gives $PA^2 = 2^2 + (2\sqrt{2})^2 = 12$, so $PA = 2\sqrt{3} = PC$. Now we consider the circular base. Since the arcs AC and CB are in the ratio $2 : 1$, then $\angle AOC : \angle COB$ is also $2 : 1$, giving $\angle COB = 60°$. Triangle BOC is therefore equilateral, as $OB = OC$, giving us that $BC = 2$. Triangle ACB is right-angled, as angles in a semicircle are 90°. Applying Pythagoras' Theorem to triangle ACB gives $AC^2 + 2^2 = 4^2$, so $AC = 2\sqrt{3}$.



As $PA = PC = AC = 2\sqrt{3}$, triangle PAC is now shown to be equilateral. The shortest distance h from A to line PC is therefore the line of symmetry of that equilateral triangle. So $h^2 + \sqrt{3}^2 = (2\sqrt{3})^2$ giving $h^2 = 9$ and therefore $h = 3$.

25. C Let O be the centre of the quarter circle, A and B be the ends of the diameter of the semicircle and M be the midpoint of AB. Let P and Q be the points on the straight edges of the quarter circle where the quarter circle is tangent to the semicircle. Let the radius of the quarter circle be 1, so $OA = 1$. Let the radius of the semicircle be r, so $MA = r$. As a tangent to a circle is perpendicular to its radius, $PM = r = QM$, and $OPMQ$ is a square. Using Pythagoras' Theorem on triangle OPM gives $OM = \sqrt{2}r$. Considering triangle OAM gives $(\sqrt{2}r)^2 + r^2 = 1^2$ so $3r^2 = 1$ and $r^2 = \frac{1}{3}$.



The area of the quarter circle is $\frac{1}{4} \times \pi \times 1^2 = \frac{\pi}{4}$. The area of the shaded semicircle is $\frac{1}{2} \times \pi r^2$ which is $\frac{1}{2} \times \pi \times \frac{1}{3}$, so $\frac{\pi}{6}$. The fraction of the quarter circle which is shaded is then $\frac{\pi/6}{\pi/4} = \frac{2}{3}$.